Homework 4 Solutions

Questions 1

The corneal shape is given by

```
In[23]:= f[r_] = Piecewise[{{8 - Sqrt[8^2 - r^2], r < 3}, {7.8 - Sqrt[7.8^2 - r^2] + c, r ≥ 3}}]
```

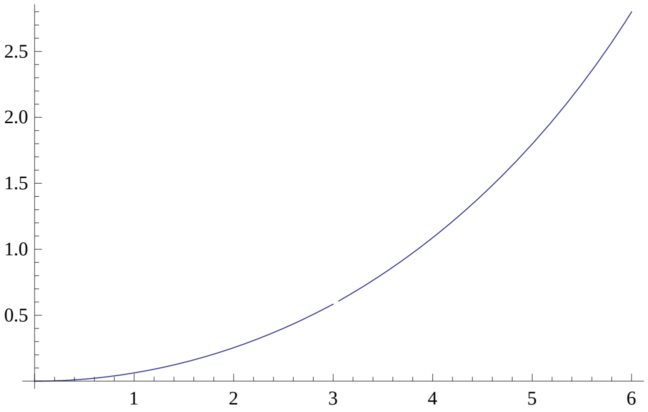
Out[23]=

$$\begin{cases} 8 - \sqrt{64 - r^2} & r < 3 \\ 7.8 + c - \sqrt{60.84 - r^2} & r \geq 3 \\ 0 & \text{True} \end{cases}$$

To find the constant C

```
In[24]:= Solve[8 - Sqrt[64 - 3^2] == f[3], c]

Out[24]= {{c → -0.0161985}}

In[27]:= c = -0.016198487095662628`

Out[27]= -0.0161985

In[30]:= Plot[f[r], {r, 0, 6}]
```

Out[30]=



The radial derivate of f is given by

```
In[31]:= dfdr[r_] = D[f[r], r]
```

Out[31]=

$$\begin{cases} \frac{1.\,r}{\sqrt{64.-1.\,r^2}} & r < 3 \\ \frac{1.\,r}{\sqrt{60.84-1.\,r^2}} & r > 3 \\ \text{Indeterminate} & \text{True} \end{cases}$$

The axial power in diopters is given by

```
In[36]:= Simplify[ϕa[r_] = 1000 * (1.3375 - 1) * dfdr[r] / (r * Sqrt[1 + dfdr[r] ^2])]
```

Out[36]=
$$\begin{cases} \text{Indeterminate} & r == 3 \\ 43.2692 \sqrt{\frac{1}{60.84-1.\, r^2}} \sqrt{60.84 - 1.\, r^2} & r > 3 \\ 42.1875 \sqrt{\frac{1}{64.-1.\, r^2}} \sqrt{64. - 1.\, r^2} & \text{True} \end{cases}$$

Note that the radicals cancel in the above expression, so that

```
In[38]:= ϕa[r_] = Piecewise[{{43.26923076923076`, r >= 3}, {42.187499999999986`, r < 3}}]
```

Out[38]=
$$\begin{cases} 43.2692 & r \geq 3 \\ 42.1875 & r < 3 \\ 0 & \text{True} \end{cases}$$

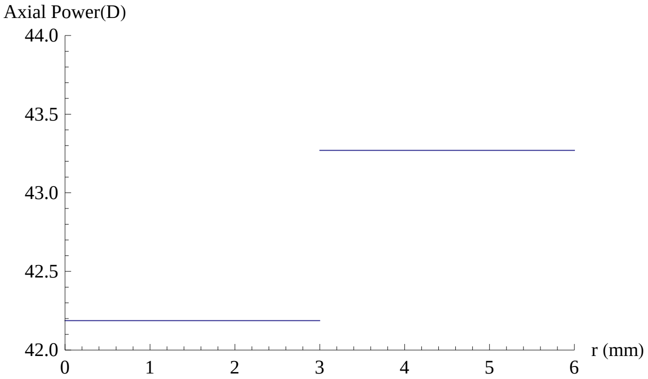
The instantaneous power is then given by

```
In[39]:= Simplify[ϕi[r_] = D[r * ϕa[r], r]]
```

Out[39]=
$$\begin{cases} 42.1875 & r < 3 \\ 43.2692 & r > 3 \\ \text{Indeterminate} & \text{True} \end{cases}$$

```
In[44]:= Plot[ϕa[r], {r, 0, 6}, PlotRange → {{0, 6}, {42, 44}}, AxesLabel → {"r  (mm)", "Axial Power(D)"}]
```

Out[44]=



The plot for the instantaneous power is identical.

Question 2

The various Zernike polynomials in the question are given by

```
In[45]:= Z4m4[ρ_, θ_] = ZernikeR[4, 4, ρ] * Sin[4 * θ]
         Z42[ρ_, θ_] = ZernikeR[4, 2, ρ] * Sin[2 * θ]
         Z22[ρ_, θ_] = ZernikeR[2, 2, ρ] * Sin[2 * θ]
```

Out[45]= $\rho^4\ \text{Sin}[4\,\theta]$

Out[46]= $\rho^2\ \left(-3 + 4\,\rho^2\right)\ \text{Sin}[2\,\theta]$

Out[47]= $\rho^2\ \text{Sin}[2\,\theta]$

For orthogonal polynomials, we expect the following integrals to be equal to zero

$$\int_0^{2\pi}\int_0^1 Z_4^{-4}(\rho,\,\theta)\,Z_4^2(\rho,\,\theta)\,\rho\,d\rho\,d\theta \quad = \quad 0 \quad \text{and} \quad \int_0^{2\pi}\int_0^1 Z_4^2(\rho,\,\theta)\,Z_2^2(\rho,\,\theta)\,\rho\,d\rho\,d\theta \quad = \quad 0$$

For the first integral, we have

$$\int_0^{2\pi}\sin 4\theta\,\sin 2\theta\,d\theta\times\int_0^1\rho^7\left(4\,\rho^2-3\right)d\rho$$

In this case the θ integral is zero

```
In[48]:=  ∫_0^(2 π) Sin[4 * θ] Sin[2 * θ] ⅆθ

Out[48]=  0
```

For the second integral, we have

$$\int_0^{2\pi}\sin 2\theta\,\sin 2\theta\,d\theta\times\int_0^1\rho^5\left(4\,\rho^2-3\right)d\rho$$

In this case, the ρ integral is zero.

```
In[51]:=  ∫_0^1 ρ^5 * (4 * ρ^2 - 3) ⅆρ

Out[51]=  0
```

Question 3

For an object at x = -95 mm and a 15 D lens, the Gaussian imaging equation says the image is formed at

```
In[52]:=  Solve[1 / Lp - 1 / (-95) == 0.015, Lp]

Out[52]=  {{Lp → 223.529}}

In[53]:=  Lp = 223.52941176470588`

Out[53]=  223.529
```

The magnification of this object is

```
In[54]:=  m = Lp / (-95)

Out[54]=  -2.35294
```

so the image of this point is formed at

```
In[55]:=  {223.539, m * -5}

Out[55]=  {223.539, 11.7647}
```

Similarly, for an object at x = -105 mm and a 15 D lens, the Gaussian imaging equation says the image is formed at

```
In[57]:= Clear[Lp]
         Solve[1 / Lp - 1 / (-105) == 0.015, Lp]

Out[58]= {{Lp → 182.609}}

In[59]:= Lp = 182.60869565217396`

Out[59]= 182.609
```

The magnification of this object is

```
In[60]:= m = Lp / (-105)

Out[60]= -1.73913
```

so the image of this point is formed at

```
In[61]:= {182.609, m * 10}

Out[61]= {182.609, -17.3913}
```

The line passing through the object points is given by

```
In[70]:= y1[x1_] = ((-5 - 10) / (-95 + 105)) * (x1 + 105) + 10

              3 (105 + x1)
Out[70]= 10 - ------------
                   2
```

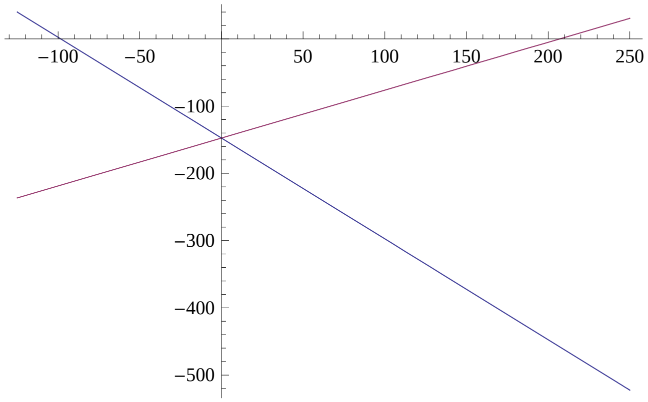
The line passing through the image points is given by

```
In[75]:= y2[x1_] =
           ((11.764705882352942` + 17.391304347826093`) / (223.52941176470588` - 182.60869565217396`)) *
             (x1 - 182.60869565217396`) - 17.391304347826093`

Out[75]= -17.3913 + 0.7125 (-182.609 + x1)
```

These lines intersect at

```
In[76]:= Solve[y1[x1] == y2[x1], x1]

                             -13
Out[76]= {{x1 → 1.02768 × 10   }}

In[77]:= Plot[{y1[x1], y2[x1]}, {x1, -125, 250}]
```

Out[77]=



Question 4

This question gives 20 random points in polar coordinates (r, θ, f (r, θ)).   We want to fit these to a second order sert of Zernike polynomials (i.e. $Z_n^m(\rho,\theta)$ for n ≤ 2) for a normalization radius $r_{\max} = 3$ mm.

```
In[79]:= polarPts = {{2.268531537, 4.525910625, -0.3406714 },
    {0.090688122, 0.394872682, 0.000822883 },
    {2.858828928, 3.114679436, -0.550426289},
    {2.205374142, 3.1876441, -0.326992507 },
    {0.509892717, 5.962904149, -0.016148879 },
    {1.666455499, 4.734426741, -0.183100928},
    {0.456445367, 6.275259981, -0.012689789},
    {0.277908028, 1.731295973, -0.003755755 },
    {0.549626698, 1.335002163, -0.018708401 },
    {2.54022574, 2.274885074, -0.430211919 },
    {2.130476538, 3.06050893, -0.305045528 },
    {1.343851106, 3.707783486, -0.119988731 },
    {1.940619109, 5.497056489, -0.250839262 },
    {0.958501009, 3.303513206, -0.060626631 },
    {2.924674713, 5.80417449, -0.574166323 },
    {2.918002861, 1.211767778, -0.565387319 },
    {1.095808545, 0.573527706, -0.079312407},
    {0.46898854, 2.508084566, -0.01338962 },
    {1.137233164, 4.803472824, -0.08454648 },
    {1.489102062, 3.19066, -0.148331149}};
```

First we construct a 20 x 6 matrix Z, where the ith row of Z is given by

$$\{Z_0^0(\rho_i,\ \theta_i) \qquad Z_1^{-1}(\rho_i,\ \theta_i) \qquad Z_1^1(\rho_i,\ \theta_i) \qquad Z_2^{-2}(\rho_i,\ \theta_i) \qquad Z_2^0(\rho_i,\ \theta_i) \qquad Z_2^2(\rho_i,\ \theta_i)\}$$

and $\rho_i = \frac{r_i}{r_{\max}}$

```
In[82]:= Z = {{1, -1.486135031, -0.280389684, 0.510347429, 0.248736096, -1.3043404},
   {1, 0.023257916, 0.055806178, 0.00158964, -1.728885259, 0.001575879},
   {1, 0.051287331, -1.905195757, -0.119672767, 1.413700963, 2.221160852},
   {1, -0.067683184, -1.468690698, 0.121746576, 0.139978614, 1.318114126},
   {1, -0.10702085, 0.322642074, -0.042289741, -1.631980386, 0.056732905},
   {1, -1.110700565, 0.024481316, -0.033302542, -0.663154554, -0.755089757},
   {1, -0.002411627, 0.304287355, -0.000898752, -1.651859791, 0.056696489},
   {1, 0.182890821, -0.02960859, -0.006632164, -1.702323861, -0.01994643},
   {1, 0.356278695, 0.085600785, 0.037351946, -1.615776502, -0.073244039},
   {1, 1.290776351, -1.096259041, -1.73304491, 0.751612598, -0.284336587},
   {1, 0.115038495, -1.415651261, -0.19945508, 0.014984272, 1.219132255},
   {1, -0.480580149, -0.756095796, 0.445028967, -1.036945795, 0.208649724},
   {1, -0.915484792, 0.914147962, -1.024974967, -0.28251576, -0.001497806},
   {1, -0.103015812, -0.630642214, 0.079566922, -1.378433704, 0.237047744},
   {1, -0.898658072, 1.730337531, -1.904455971, 1.560278393, 1.338941088},
   {1, 1.821297941, 0.6835225, 1.524682591, 1.545274415, -1.745214401},
   {1, 0.396389502, 0.613646997, 0.297910895, -1.26986407, 0.134377783},
   {1, 0.185086565, -0.251989346, -0.057121908, -1.647391917, 0.017906746},
   {1, -0.755012686, 0.068960267, -0.063767415, -1.234259687, -0.34616718},
   {1, -0.048691314, -0.991539892, 0.059129924, -0.878563546, 0.600602951}};
```

Next, construct a 20 element vector f where the ith element is f(r_i,θ_i)

```
In[85]:= f = {-0.3406714, 0.000822883, -0.550426289, -0.326992507, -0.016148879,
     -0.183100928, -0.012689789, -0.003755755, -0.018708401, -0.430211919,
     -0.305045528, -0.119988731, -0.250839262, -0.060626631, -0.574166323,
     -0.565387319, -0.079312407, -0.01338962, -0.08454648, -0.148331149};
```

We now have the matrix equation

Za = f

where a is the vector of coefficients which describe the weights of each of the Zernike terms. The goal of the problem then is to solve for a. Since Z is not square, we cannot simply multiply each side of the equation by Z^{-1}. Instead, we need to do a least squares solution

a = $\left(Z^T Z\right)^{-1} Z^T$f

```
In[86]:= a = Inverse[Transpose[Z].Z].Transpose[Z].f

Out[86]= {-0.3, 3.81986 × 10^-11, 9.82106 × 10^-12, 1.69294 × 10^-10, -0.174, -0.002}
```

```
In[88]:= BarChart[a, ChartLabels → {"a00", "a1-1", "a11", "a2-2", "a20", "a22"}]
```

Out[88]=