OPTI 435/535 Homework #5 Due: December 5, 2011

Undergrads do problems 1-3. Grads do all problems

1. The Munnerlyn formula describes the shape pf the post-LASIK cornea. Over the central optical zone, the cornea can be approximated as a sphere of radius R1. Outside the optical zone, the cornea is a sphere of radius R2. For R1 = 8 mm, R2 = 7.8 mm and an optical zone diameter of 6 mm, the sag of the cornea is given by

$$f(r) = \begin{cases} 8 - \sqrt{8^2 - r^2} & \text{for } r < 3.0\text{ mm} \\ 7.8 - \sqrt{7.8^2 - r^2} + C & \text{for } r \geq 3.0\text{ mm} \end{cases}$$

(a) Find the constant C such that the cornea is continuous at r = 3.0 mm.

The corneal shape is given by

```
In[23]:= f[r_] = Piecewise[{{8 - Sqrt[8^2 - r^2], r < 3}, {7.8 - Sqrt[7.8^2 - r^2] + c, r ≥ 3}}]
```

$$\text{Out[23]=} \begin{cases} 8 - \sqrt{64 - r^2} & r < 3 \\ 7.8 + c - \sqrt{60.84 - r^2} & r \geq 3 \\ 0 & \text{True} \end{cases}$$

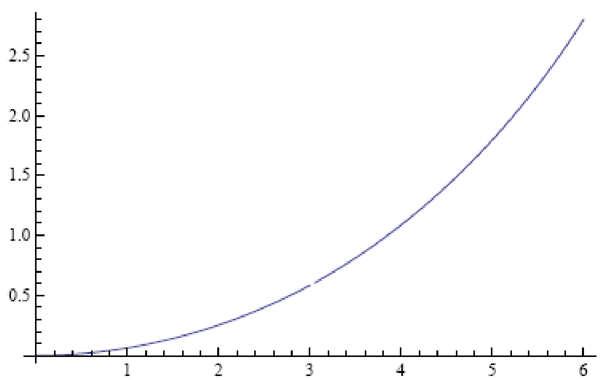
To find the constant C

```
In[24]:= Solve[8 - Sqrt[64 - 3^2] == f[3], c]
Out[24]= {{c → -0.0161985}}

In[27]:= c = -0.016198487095662628`
Out[27]= -0.0161985

In[30]:= Plot[f[r], {r, 0, 6}]
```

Out[30]=



(b) What is the axial power of the cornea as a function of r? Assume $n_k = 1.3375$.

The radial derivate of f is given by

```
In[31]:= dfdr[r_] = D[f[r], r]
```

$$\text{Out[31]=} \begin{cases} \dfrac{1.\,r}{\sqrt{64.-1.\,r^2}} & r < 3 \\ \dfrac{1.\,r}{\sqrt{60.84-1.\,r^2}} & r > 3 \\ \text{Indeterminate} & \text{True} \end{cases}$$

The axial power in diopters is given by

```
In[36]:= Simplify[ϕa[r_] = 1000 * (1.3375 - 1) * dfdr[r] / (r * Sqrt[1 + dfdr[r]^2])]
```

Out[36]=

$$\begin{cases} \text{Indeterminate} & r == 3 \\ 43.2692\sqrt{\frac{1}{60.84-1.\,r^2}}\sqrt{60.84-1.\,r^2} & r > 3 \\ 42.1875\sqrt{\frac{1}{64.-1.\,r^2}}\sqrt{64.-1.\,r^2} & \text{True} \end{cases}$$

Note that the radicals cancel in the above expression, so that

```
In[38]:= ϕa[r_] = Piecewise[{{43.26923076923076`, r >= 3}, {42.187499999999986`, r < 3}}]
```

Out[38]=

$$\begin{cases} 43.2692 & r \geq 3 \\ 42.1875 & r < 3 \\ 0 & \text{True} \end{cases}$$

(c) What is the instantaneous power of the cornea as a function of r?

The instantaneous power is then given by

```
In[39]:= Simplify[ϕi[r_] = D[r * ϕa[r], r]]
```

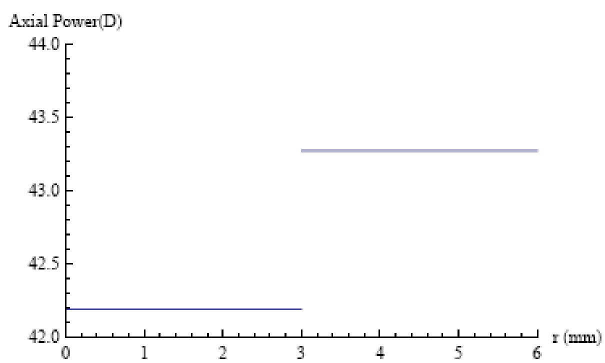
Out[39]=

$$\begin{cases} 42.1875 & r < 3 \\ 43.2692 & r > 3 \\ \text{Indeterminate} & \text{True} \end{cases}$$

(d) Plot the results of parts b and c.

```
In[44]:= Plot[ϕa[r], {r, 0, 6}, PlotRange → {{0, 6}, {42, 44}}, AxesLabel → {"r (mm)", "Axial Power(D)"}]
```

Out[44]=



The plot for the instantaneous power is identical.

2. Suppose we have the following Scheimpflug system. A 15D thin lens is located at z = 0 in the x-y plane. Two object points are located at (0, 10 mm, -105 mm) and (0, -5 mm, -95 mm), respectively.

(a) Where are the image points formed?

For an object at x = -95 mm and a 15 D lens, the Gaussian imaging equation says the image is formed at

```
In[52]:= Solve[1 / Lp - 1 / (-95) == 0.015, Lp]

Out[52]= {{Lp → 223.529}}

In[53]:= Lp = 223.52941176470588`

Out[53]= 223.529
```

The magnification of this object is

```
In[54]:= m = Lp / (-95)

Out[54]= -2.35294
```

so the image of this point is formed at

```
In[55]:= {223.539, m * -5}

Out[55]= {223.539, 11.7647}
```

Similarly, for an object at x = -105 mm and a 15 D lens, the Gaussian imaging equation says the image is formed at

```
In[57]:= Clear[Lp]
         Solve[1 / Lp - 1 / (-105) == 0.015, Lp]

Out[58]= {{Lp → 182.609}}

In[59]:= Lp = 182.60869565217396`

Out[59]= 182.609
```

The magnification of this object is

```
In[60]:= m = Lp / (-105)

Out[60]= -1.73913
```

so the image of this point is formed at

```
In[61]:= {182.609, m * 10}

Out[61]= {182.609, -17.3913}
```

(b) What is the equation of the line that passes through the two object points?

The line passing through the object points is given by

```
In[70]:= y1[x1_] = ((-5 - 10) / (-95 + 105)) * (x1 + 105) + 10
```

Out[70]= $10 - \frac{3\,(105 + x1)}{2}$

(c) What is the equation of the line that passes through the two image points?

The line passing through the image points is given by

```
In[75]:= y2[x1_] =
  ((11.764705882352942` + 17.391304347826093`) / (223.52941176470588` - 182.60869565217396`)) *
    (x1 - 182.60869565217396`) - 17.391304347826093`

Out[75]= -17.3913 + 0.7125 (-182.609 + x1)
```

(d) Where do these two lines intersect?

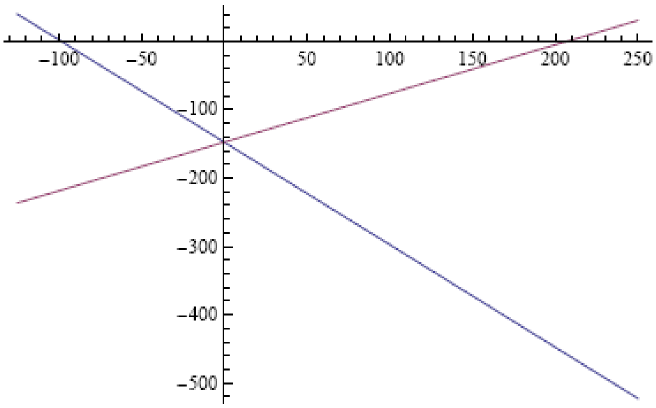
These lines intersect at

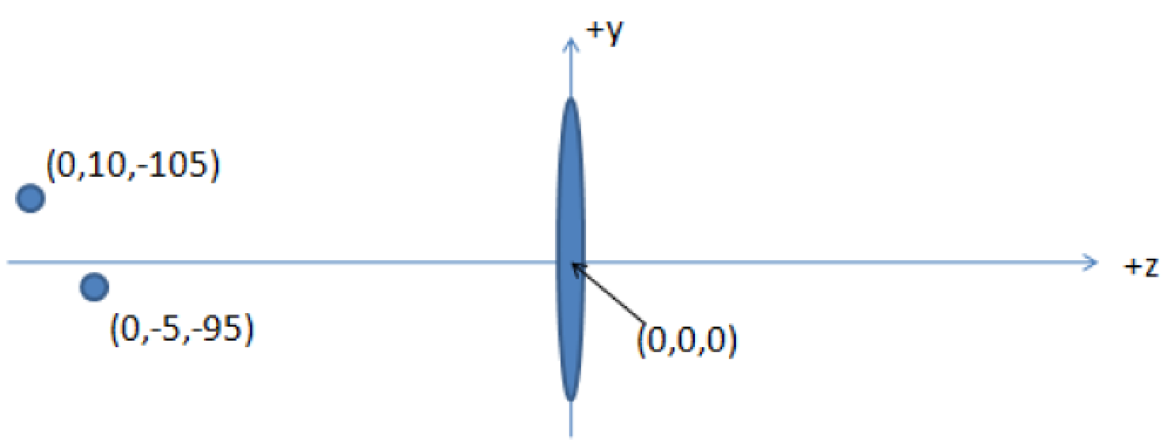
```
In[76]:= Solve[y1[x1] == y2[x1], x1]

Out[76]= {{x1 → 1.02768 × 10^-13}}

In[77]:= Plot[{y1[x1], y2[x1]}, {x1, -125, 250}]
```

Out[77]=





3. An electronic copy (ColorimetryData.txt) of the data for the various colorimetric functions for this problem is available on the web site. The columns of the data are: the wavelength in 10 nm steps, a spectral reflectance for a white patch, a spectral reflectance for an unknown color patch, the spectral distribution for Illuminant C, and the CIE 1931 2° color matching functions $\overline{x}(\lambda)$, $\overline{y}(\lambda)$, and $\overline{z}(\lambda)$ (xbar, ybar, zbar). In the notes, these are labeled x', y' and z', but the bar notation is more universal.

(a) Plot the x,y chromaticity coordinates for spectrally pure colors (i.e. $P(\lambda) = \delta(\lambda - \lambda_o)$ where λ_o ranges from 380 nm to 780 nm in 10 nm steps).

(b) Calculate the Tristimulus values X, Y and Z and X_w, Y_w and Z_w for the color patch and the white patch respectively. Assume that Illuminant C is used to illuminate these patches for this calculation.

(c) Calculate the x,y chromaticity coordinates for the color patch and the white patch and plot them on the plot from question 1.

(d) What are the approximate values of the Dominant Wavelength, the Complementary Color and the
Excitation Purity assuming the white patch represents the White Point of the system?

(e) Calculate L*, a* and b* in the CIELAB color space for the color patch and the white patch. Again, assume the white patch is the white point for the system.

(f) Calculate ΔE between the white patch and the color patch.

See HW5_11 Q3 Solutions.xls

*********************************Grads Only ***********************************

4. Fit the points below to a 2nd order Zernike expansion (i.e. $n \le 2$) for a normalization radius of 3 mm.

This question gives 20 random points in polar coordinates (r, θ, f (r, θ)). We want to fit these to a second order sert of Zernike polynomials (i.e. $Z_n^m(\rho,\theta)$ for $n \le 2$) for a normalization radius $r_{max} = 3$ mm.

```
In[79]:= polarPts = {{2.268531537, 4.525910625, -0.3406714 },
   {0.090688122, 0.394872682, 0.000822883 },
   {2.858828928, 3.114679436, -0.550426289},
   {2.205374142, 3.1876441, -0.326992507 },
   {0.509892717, 5.962904149, -0.016148879 },
   {1.666455499, 4.734426741, -0.183100928},
   {0.456445367, 6.275259981, -0.012689789},
   {0.277908028, 1.731295973, -0.003755755 },
   {0.549626698, 1.335002163, -0.018708401 },
   {2.54022574, 2.274885074, -0.430211919 },
   {2.130476538, 3.06050893, -0.305045528 },
   {1.343851106, 3.707783486, -0.119988731 },
   {1.940619109, 5.497056489, -0.250839262 },
   {0.958501009, 3.303513206, -0.060626631 },
   {2.924674713, 5.80417449, -0.574166323 },
   {2.918002861, 1.211767778, -0.565387319 },
   {1.095808545, 0.573527706, -0.079312407},
   {0.46898854, 2.508084566, -0.01338962 },
   {1.137233164, 4.803472824, -0.08454648 },
   {1.489102062, 3.19066, -0.148331149}};
```

First we construct a 20 x 6 matrix Z, where the ith row of Z is given by

$\{Z_0^0(\rho_i,\theta_i) \quad Z_1^{-1}(\rho_i,\theta_i) \quad Z_1^1(\rho_i,\theta_i) \quad Z_2^{-2}(\rho_i,\theta_i) \quad Z_2^0(\rho_i,\theta_i) \quad Z_2^2(\rho_i,\theta_i)\}$

and $\rho_i = \frac{r_i}{r_{max}}$

```
In[82]:= Z = {{1, -1.486135031, -0.280389684, 0.510347429, 0.248736096, -1.3043404},
   {1, 0.023257916, 0.055806178, 0.00158964, -1.728885259, 0.001575879},
   {1, 0.051287331, -1.905195757, -0.119672767, 1.413700963, 2.221160852},
   {1, -0.067683184, -1.468690698, 0.121746576, 0.139978614, 1.318114126},
   {1, -0.10702085, 0.322642074, -0.042289741, -1.631980386, 0.056732905},
   {1, -1.110700565, 0.024481316, -0.033302542, -0.663154554, -0.755089757},
   {1, -0.002411627, 0.304287355, -0.000898752, -1.651859791, 0.056696489},
   {1, 0.182890821, -0.02960859, -0.006632164, -1.702323861, -0.01994643},
   {1, 0.356278695, 0.085600785, 0.037351946, -1.615776502, -0.073244039},
   {1, 1.290776351, -1.096259041, -1.73304491, 0.751612598, -0.284336587},
   {1, 0.115038495, -1.415651261, -0.19945508, 0.014984272, 1.219132255},
   {1, -0.480580149, -0.756095796, 0.445028967, -1.036945795, 0.208649724},
   {1, -0.915484792, 0.914147962, -1.024974967, -0.28251576, -0.001497806},
   {1, -0.103015812, -0.630642214, 0.079566922, -1.378433704, 0.237047744},
   {1, -0.898658072, 1.730337531, -1.904455971, 1.560278393, 1.338941088},
   {1, 1.821297941, 0.6835225, 1.524682591, 1.545274415, -1.745214401},
   {1, 0.396389502, 0.613646997, 0.297910895, -1.26986407, 0.134377783},
   {1, 0.185086565, -0.251989346, -0.057121908, -1.647391917, 0.017906746},
   {1, -0.755012686, 0.068960267, -0.063767415, -1.234259687, -0.34616718},
   {1, -0.048691314, -0.991539892, 0.059129924, -0.878563546, 0.600602951}};
```

Next, construct a 20 element vector f where the ith element is f(r_i,θ_i)

```
In[85]:= f = {-0.3406714, 0.000822883, -0.550426289, -0.326992507, -0.016148879,
     -0.183100928, -0.012689789, -0.003755755, -0.018708401, -0.430211919,
     -0.305045528, -0.119988731, -0.250839262, -0.060626631, -0.574166323,
     -0.565387319, -0.079312407, -0.01338962, -0.08454648, -0.148331149};
```

We now have the matrix equation

Za = f

where a is the vector of coefficients which describe the weights of each of the Zernike terms. The goal of the problem then is to solve for a. Since Z is not square, we cannot simply multiply each side of the equation by Z^{-1}. Instead, we need to do a least squares solution

$$a = \left(Z^T Z\right)^{-1} Z^T f$$

```
In[86]:= a = Inverse[Transpose[Z].Z].Transpose[Z].f
Out[86]= {-0.3, 3.81986 × 10^-11, 9.82106 × 10^-12, 1.69294 × 10^-10, -0.174, -0.002}
```

```
In[88]:= BarChart[a, ChartLabels → {"a00", "a1-1", "a11", "a2-2", "a20", "a22"}]
```

Out[88]=