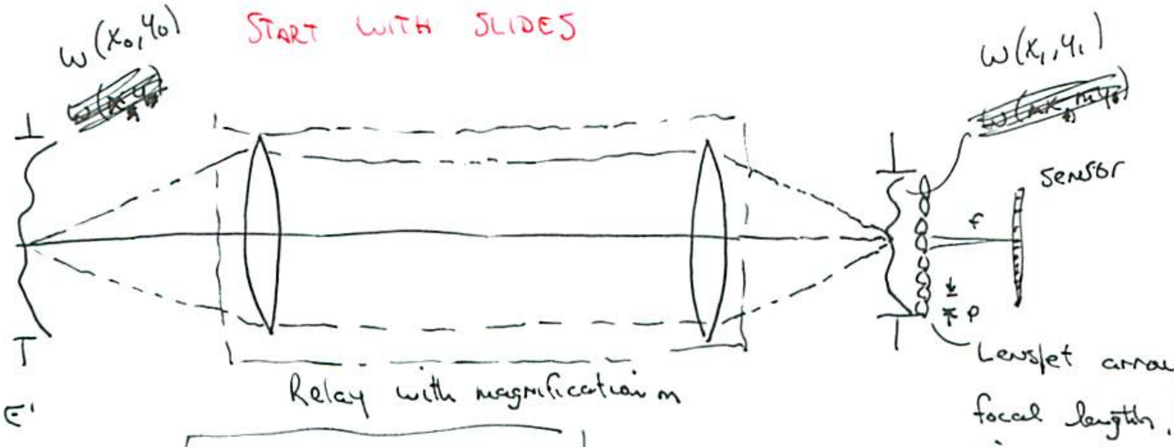
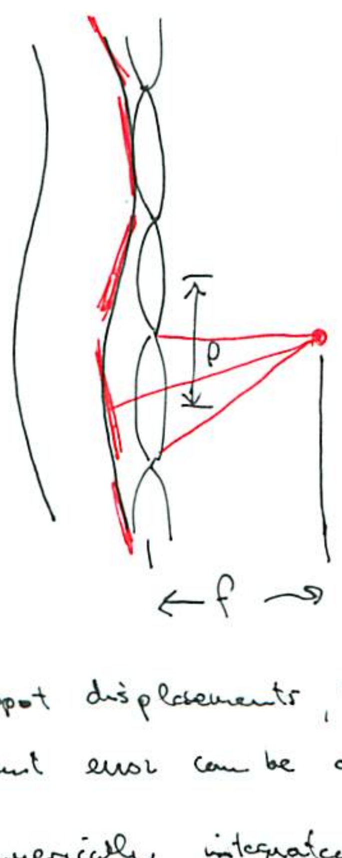
3.2.5 SHACK-HARTMANN SENSOR

START WITH SLIDES

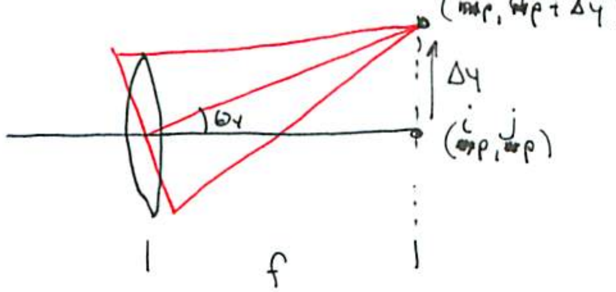


$$\boxed{x_1 = m x_0 \qquad y_1 = m y_0}$$

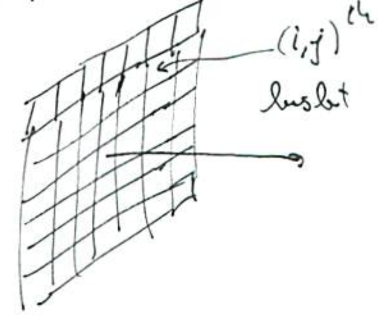
Wavefront from exit pupil is relayed to lenslet array. If relay has magnification m, then a wavefront error $W(x_0, y_0)$ gets projected onto the lenslet array as $W(x_1, y_1)$. This means the wavefront is compressed in the transverse direction, but the phase remains the same.



Assume that the wavefront is slowly varying over the aperture of a given lenslet. We can assume that the wavefront over a given lenslet is given by a tilted plane wave. The tilt is given by the local wavefront slope. A plane wave is focused to the rear focal plane of the lenslet. The focal spot, however, is displaced from the optical axis of the lenslet due to the tilt. By measuring the spot displacements, the gradient (i.e. x and y derivatives) of the wavefront error can be obtained. In turn, the gradient can be numerically integrated to recover the wavefront error.



Consider the $(i,j)^{th}$ lenslet in the array. It is centered at a point (i_p, j_p)



$$\tan \theta_y = \frac{\Delta y}{f}$$

but $$\tan \theta_y = -\frac{dw(ip, jp)}{dy}$$

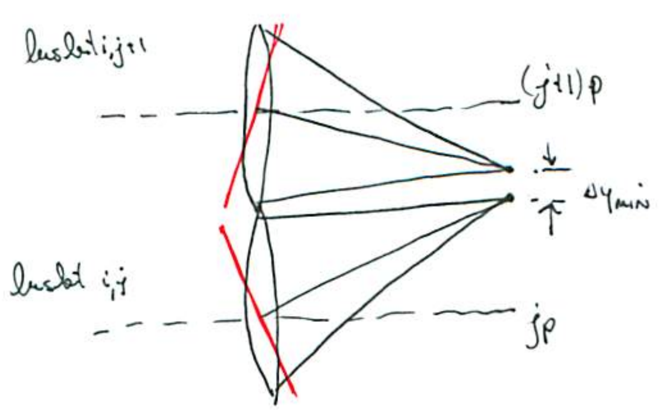
so

$$\boxed{\begin{aligned} \Delta y_{ij} &= -f \frac{dw}{dy_1}(ip, jp) \\ \text{Similarly} \quad \Delta x_{ij} &= -f \frac{dw}{dx_1}(ip, jp) \end{aligned}}$$

Practically, we don't get perfect point but instead some finite size to the focal spot. In this case, the spot centroid is used. There is a maximum change in slope that can occur between two adjacent lenslets.

Dynamic Range
MAXIMUM SLOPE CHANGE BETWEEN LENSLETS

Spots can merge or cross over if slope change is too big between lenslets



Want

$$(j+1)p + \Delta y_{i,j+1} - \left[jp + \Delta y_{i,j} \right] \geq \Delta y_{min}$$

where Δy_{min} is some minimum separation between adjacent spots

So,

$$p + \Delta y_{i,j+1} - \Delta y_{ij} \geq \Delta y_{min}$$

$$p - f\,\frac{\partial w}{\partial y}(ip,(j+1)p) + f\,\frac{\partial w}{\partial y}(ip,jp) \geq \Delta y_{min}$$

$$\boxed{\frac{\partial w}{\partial y_1}(ip,(j+1)p) - \frac{\partial w}{\partial y_1}(ip,jp) \leq \frac{p - \Delta y_{min}}{f}}$$

MAXIMUM SLOPE CHANGE BETWEEN LENSLETS

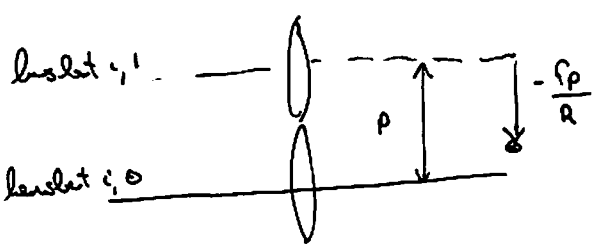
$$\frac{\frac{\partial w}{\partial y_1}(ip,(j+1)p) - \frac{\partial w}{\partial y_1}(ip,jp)}{p} \approx \boxed{\frac{\partial^2 w}{\partial y_1^2}(ip,jp) \leq \frac{p - \Delta y_{min}}{pf}}$$

MAXIMUM WAVEFRONT CURVATURE

Simple Example w 1-D

$$w = \frac{y_1^2}{2R} \Rightarrow \frac{\partial w}{\partial y_1} = \frac{y_1}{R} \Rightarrow \frac{\partial^2 w}{\partial y_1^2} = \frac{1}{R}$$

Suppose $\Delta y_{min} = 0$



$$\Delta y_{i,0} = 0$$

$$\Delta y_{i,1} = -\frac{fp}{R}$$

Maximum Slope Change Says

$$\frac{p}{R} - 0 \leq \frac{p}{f} \Rightarrow \frac{1}{R} \leq \frac{1}{f} \Rightarrow R \geq f$$

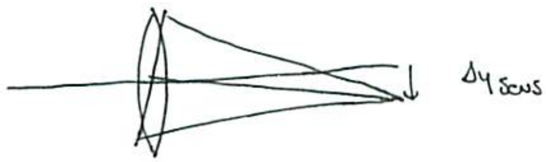
means $\Delta y_{i,1} \leq p$

MAXIMUM WAVEFRONT CURVATURE Says

$$\frac{1}{R} \leq \frac{1}{f}$$ same thing

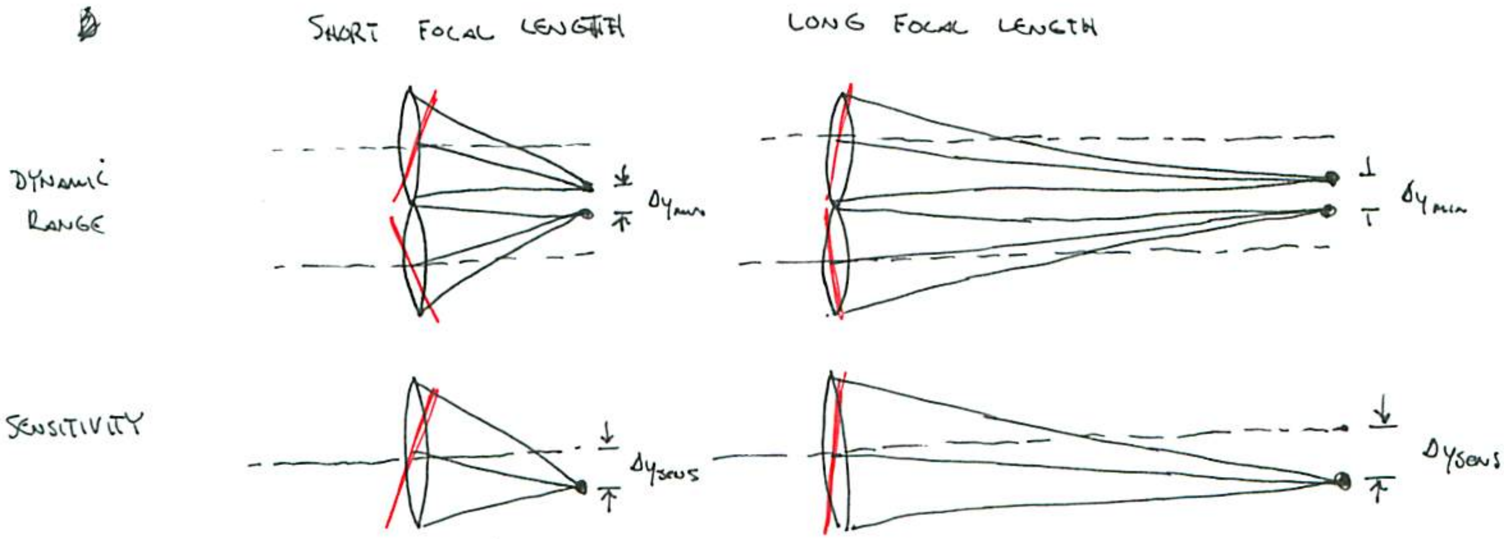
Practically, Δy_{min} will be limited by the pixel size on the detector or the size of the focal spot (like Rayleigh criteria how close can these spots get and still be resolved?)

Sensitivity - what's the smallest absolute slope that can be measured?



$$\Delta y_{sens} = -f \frac{d\omega_{min}}{dy_1}$$

$$\boxed{\left| \frac{d\omega_{min}}{dy} \right| = \frac{\Delta y_{sens}}{f}}$$



sensitivity ↓
dynamic range ↑

sensitivity ↑
dynamic range ↓

Effect of Relay System

$\omega(x_0, y_0) = \omega(x_1, y_1)$ where $x_1 = m x_0$; $y_1 = m y_0$ m = relay magnification

$$\frac{dx_1}{dx_0} = m \; ; \; \frac{dy_1}{dy_0} = m$$

chain Rule

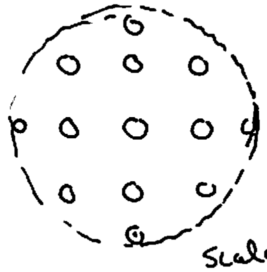
$$\frac{d\omega}{dx_0} = \frac{d\omega}{dx_1}\frac{dx_1}{dx_0} \Rightarrow \boxed{\begin{aligned} \frac{d\omega}{dx_1} &= \frac{1}{m}\frac{d\omega}{dx_0} \\ \frac{d\omega}{dy_1} &= \frac{1}{m}\frac{d\omega}{dy_0} \end{aligned}}$$

Slope at lenslet → $\frac{d\omega}{dy_1}$; slope at E' → $\frac{d\omega}{dy_0}$

For $m < 1$, slope is steeper at lenslet array than at E'

Dynamic Range ↓

SHACK HARTMANN SPOT PATTERN



$$\mathrm{cyl}\left(\frac{r_1}{m D_{E'}}\right) \sum_{i,j} \left[\delta\left(x_1 - ip + f\frac{dW}{dx_1}(ip,jp),\ y_1 - jp + f\frac{dW}{dy_1}(ip,jp)\right) * B(x_1,y_1) \right]$$

scaled exit pupil — spot shifts due to aberration — blur due to source size and diffraction

$$r_1^2 = x_1^2 + y_1^2$$

$$\mathrm{cyl}\left(\frac{r}{m D_{E'}}\right) \sum_{i,j} B\left(x_1 - ip + f\frac{dW}{dx_1}(ip,jp),\ y_1 - jp + f\frac{dW}{dy_1}(ip,jp)\right)$$

Centroid of $B(x_1,y_1)$ is a (d_x, d_y) where

$$d_x = \frac{\int x_1 B(x_1,y_1)\,dx_1 dy_1}{\int B(x_1,y_1)\,dx_1 dy_1} \qquad d_y = \frac{\int y_1 B(x_1,y_1)\,dx_1 dy_1}{\int B(x_1,y_1)\,dx_1 dy_1}$$

Centroids of the spots are given by

$$\mathrm{cyl}\left(\frac{r_1}{m D_{E'}}\right) \sum_{i,j} \delta\left(x_1 - ip + f\frac{dW}{dx_1}(ip,jp) - d_x,\ y_1 - jp + f\frac{dW}{dy_1}(ip,jp) - d_y\right)$$

Centroid of $B(x_1,y_1)$ shifts entire pattern by constant amount.
The preceding is only true for ~~interior~~ lenslets interior to the cyl() function. Spots that are clipped by the cyl() function will have displaced centroids and induce slight errors.

Example $\quad W(x_0,y_0) = W_{020}(x_0^2 + y_0^2) \Rightarrow$ Defocus in Exit Pupil

$$W(x_1,y_1) = \frac{W_{020}}{m^2}(x_1^2 + y_1^2)$$

$$\frac{dW}{dx_1} = \frac{2W_{020}}{m^2} x_1 \qquad \frac{dW}{dy_1} = \frac{2W_{020}}{m^2} y_1$$

SPOT PATTERN

$$\mathrm{cyl}\left(\frac{r_1}{mD_{e'}}\right)\sum_{ij}\delta\left(x_1-\left(1-\frac{2W_{020}}{m^2}f\right)ip-d_x,\; y_1-\left(1-\frac{2W_{020}}{m^2}f\right)jp-d_y\right)$$

$\left[1-\frac{2W_{020}}{m^2}f\right]p$ Uniformly spaced grid

when $W_{020}=0$ spots ~~fall~~ separation $=p$

when $W_{020}>0$ spots uniform but compress

when $W_{020}<0$ spots uniform but expand

$\boxed{3.2.5.1}$ Fitting Shack-Hartmann Data to Zernike Polynomials

STEPS TO MEASURING A WAVEFRONT WITH A SHACK HARTMANN SYSTEM

1. Calibrate system with a perfect plane wave
   This gives a set of spots for the case where $W(x_1,y_1)=0$
   Only need to do this once
2. Measure the test system
   This gives a set of spots that are displaced by the aberrations of the system
3. Measure the distance between the ideal and aberrated spots to get
   $\{\Delta x_{ij}, \Delta y_{ij}\}_{ij}$
4. Convert ~~this~~ this set to $\left\{ \frac{dW}{dx_1}^{(ip,jp)}, \frac{dW}{dy}^{(ip,jp)} \right\}_{ij}$ with eqs. from pg. $\boxed{129}$

Want to integrate these slopes to get $W(x_1,y_1)$ back

One way to do this is to fit to Zernike polynomials

Before we did $W(\bar{x},\bar{y}) = \sum a_{nm} Z_n^m(\bar{x},\bar{y})$

$$\frac{dW(\bar{x},\bar{y})}{d\bar{x}} = \sum a_{nm}\frac{dZ_n^m(\bar{x},\bar{y})}{d\bar{x}} \qquad \frac{dW(\bar{x},\bar{y})}{d\bar{y}} = \sum a_{nm}\frac{dZ_n^m(\bar{x},\bar{y})}{d\bar{y}}$$